\[ f(x) = x^5 - 5x^3 + 4x \]

1. Fundamental Theorem of Algebra:

5 real zeroes

or

3 real zeroes and 2 non-real solutions

or

1 real zero and 4 non-real solutions

*Remember: Non-real solutions must occur in pairs*

2. Descartes' Rule of Signs:

\[ f(x) = x^5 - 5x^3 + 4x \]

\[ 1 \quad 2 \text{ sign changes} \]

There are two or zero positive **real** zeroes.

\[ f(-x) = -x^5 + 5x^3 - 4x \]

\[ 1 \quad 2 \text{ sign changes} \]

There are two or zero negative **real** zeroes.

Now, if you're feeling really observant, you might notice that we must have an odd number of real solutions, but we also must have an even (or zero) number of intercepts. How is this possible?

Simple: zero is an intercept, and it's neither positive nor negative. It won't be counted by Descartes' Rule. It's also the \(y\) – intercept.

\[ f(0) = 0^5 - 5(0)^3 + 4(0) = 0 \]

So, our first solution is \((0, 0)\), the origin.
3. Finding Positive Real Zeroes:

We’ll start with $x = 1$.

\[
\begin{array}{c|cccc}
1 & 1 & 0 & -5 & 0 & 4 \\
\hline
& 1 & 1 & -4 & -4 \\
1 & 1 & -4 & -4 & 0 \\
\end{array}
\]

…which is our first zero. Applying Descartes' Rule to the result, we see only one sign change (zero doesn’t count), so we see we have one zero left.

We then try $x = 2$.

\[
\begin{array}{c|cccc}
2 & 1 & 0 & -5 & 0 & 4 \\
\hline
& 2 & 4 & -2 & -4 \\
1 & 2 & -1 & -2 & 0 \\
\end{array}
\]

Which is our next zero, and Descartes' Rule tells us we’re done with the positive $x$-axis, since there could only be 2 or 0 intercepts.

4. Finding Negative Real Zeroes:

First, we set up our synthetic division for $f(-x)$. Then we use positive numbers, knowing that they represent positions on the negative $x$–axis when used with $f(-x)$. We’ll start with $1$.

\[
\begin{array}{c|cccc}
1 & -1 & 0 & 5 & 0 & -4 \\
\hline
& -1 & 1 & -1 & 4 & 4 \\
-1 & 1 & 4 & 4 & 0 \\
\end{array}
\]

…and the remainder of zero tells us that $x = -1$ is our first negative real intercept.

We then try $2$.

\[
\begin{array}{c|cccc}
2 & -1 & 0 & 5 & 0 & -4 \\
\hline
& -2 & -4 & 2 & 4 \\
-1 & -2 & 1 & 2 & 0 \\
\end{array}
\]

Which is our next zero, and Descartes' Rule tells us we’re done with the positive $x$-axis, since there could only be 2 or 0 intercepts.
Normally, a table of integer $x$ – values would help us with the graph. Here, the table is less useful...

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

So we can’t see any positive or negative values of $y$ to guide us. Still, we know the end behavior is “positive ballerina.”

\[
\lim_{x \to -\infty} f(x) = -\infty
\]
\[
\lim_{x \to \infty} f(x) = \infty
\]

So the graph should look something like:

Keep in mind, you’ll have no idea how high the extrema are unless you plug in values between the zeroes. You won’t be graded on that. This year.
EASTER EGG:

The preceding problem had an Easter Egg solution:

\[
\begin{align*}
f(x) &= x^5 - 5x^3 + 4x \\
f(x) &= x(x^4 - 5x^2 + 4) \\
f(x) &= x(x^2 - 1)(x^2 - 4) \\
f(x) &= x(x - 1)(x + 1)(x - 2)(x + 2)
\end{align*}
\]

So the zeroes are 0, ±1, ±2. The end behavior is “ballerina.” Graph accordingly.
\[ f(x) = 2x^4 - 18x^3 + 54x^2 - 64x + 24 \]

1. Fundamental Theorem: This polynomial has 4, 2, or 0 real zeroes.

2. Descartes’ Rule of Signs:

\[ f(x) = 2x^4 - 18x^3 + 54x^2 - 64x + 24 \]

This polynomial has 4, 2, or 0 positive zeroes.

\[ f(-x) = 2x^4 + 18x^3 + 54x^2 + 64x + 24 \]

This polynomial has 0 negative zeroes.

3. \( y \)-intercept: \( f(0) = 24 \)

So we start looking for zeroes, keeping track of the value of function each time (using the Remainder Theorem).

\[
\begin{array}{c|cccc}
1 & 2 & -18 & 54 & -64 & 24 \\
2 & 2 & -16 & 38 & -26 & \\
\hline
2 & -16 & 38 & -26 & -2 & \\
\end{array}
\]

1 is not a zero, but notice two things: the number of sign changes in our answer is now 3. So we just passed a zero. Also, the remainder (which is also the \( y \)-value) tells us that the function went from (0, 24) to (1, -2). So the function passed through the \( x \)-axis. Our first zero in between 0 and 1.

Looking at our answer, we have 3 sign changes left. Which makes sense. We were looking for 4 or 2 zeroes, and we found one. Now we have 3 or 1 zero left.

\[
\begin{array}{c|cccc}
2 & 2 & -18 & 54 & -64 & 24 \\
2 & 4 & -28 & 52 & -24 & \\
\hline
2 & -14 & 26 & -12 & 0 & \\
\end{array}
\]

2 is a zero. But notice the number of sign changes in our answer remains the same. How can we have found a zero and still be looking for 3 or 1 more zeroes?

One possibility is that the 2 is actually a zero which occurs twice. Then perhaps we went from three zeroes left to one zero left. We can check this by seeing repeating the synthetic division using our answer above.
So, 2 occurs twice as zero (in other words \((x - 2)\) occurs twice as a factor).

If we look at the answer above, we can see that our remaining zeroes will be the factors of \(2x^2 - 10 + 6\) or \(x^3 - 5x + 3\). Descartes' Rule of signs tells us that we have 2 more sign changes, and thus two or zero more \(x\) - intercepts. Since we can't easily "reverse FOIL" this quadratic, we know that we're looking two non-integer zeroes.

You could use the quadratic formula on the answer from above:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{5 \pm \sqrt{25 - 12}}{2}$$
$$x = \frac{5 \pm \sqrt{13}}{2}$$

$$x = \frac{5 + \sqrt{13}}{2}$$
$$\frac{5 + \sqrt{9}}{2} < \frac{5 + \sqrt{13}}{2} < \frac{5 + \sqrt{16}}{2}$$
$$\frac{5 + 3}{2} < \frac{5 + \sqrt{13}}{2} < \frac{5 + 4}{2}$$
$$8 < \frac{5 + \sqrt{13}}{2} < 9$$
$$4 < \frac{5 + \sqrt{13}}{2} < 4 \frac{1}{2}$$

$$\frac{5 - \sqrt{16}}{2} < \frac{5 - \sqrt{13}}{2} < \frac{5 - \sqrt{9}}{2}$$
$$\frac{5 - 4}{2} < \frac{5 - \sqrt{13}}{2} < \frac{5 - 3}{2}$$
$$1 < \frac{5 + \sqrt{13}}{2} < 2$$
$$\frac{1}{2} < \frac{5 + \sqrt{13}}{2} < 1$$

This tells me that my remaining zeroes are between -1/2 and -1 and another one is between -4 and -4 1/2.
If you decide that you hate the quadratic formula and/or like having a nice table of values, you would continue with synthetic division on the original polynomial after determining that 2 occurred twice as a zero.

\[
\begin{array}{cccccc}
3 & 2 & -18 & 54 & -64 & 24 \\
 & 6 & -36 & 54 & -30 \\
\hline
2 & -12 & 18 & -10 & -6 \\
\end{array}
\]

Now, three is not a zero. And our remainder is still negative. Let’s look at a table:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-6</td>
</tr>
</tbody>
</table>

This reminds us of the idea of **multiplicity**. Multiplicity means that zeroes which occur an odd number of times pass through the \( x \)-axis. Zeroes which occur an even number of times turn around without crossing the \( x \)-axis.

Since \( y \) is negative at \( x = 1 \), zero at \( x = 2 \), and is still negative at \( x = 3 \), we can safely assume that the function turns around \( x = 2 \) without crossing the \( x = 2 \).

We still have three sign changes in our answer, meaning three or 1 remaining zeroes. Since we have found three zeroes out of a possible four, we know that we are only looking for one more.

\[
\begin{array}{cccccc}
4 & 2 & -18 & 54 & -64 & 24 \\
 & 8 & -40 & 56 & -32 \\
\hline
2 & -10 & 14 & -8 & -8 \\
\end{array}
\]

At \( x = 4 \), we still have a negative \( y \) value, and we still have 3 sign changes.

\[
\begin{array}{cccccc}
5 & 2 & -18 & 54 & -64 & 24 \\
 & 10 & -40 & 70 & 30 \\
\hline
2 & -8 & 14 & 6 & 54 \\
\end{array}
\]

At \( x = 5 \), we have a positive \( y \) value, and we still have 2 sign changes, meaning or 2 or zero remaining intercepts. Since we were only looking for one more zero, we’ve found it, between 4 and 5.
So, to finish our table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-6</td>
</tr>
<tr>
<td>4</td>
<td>-8</td>
</tr>
<tr>
<td>5</td>
<td>54</td>
</tr>
</tbody>
</table>

So we have an intercept between 0 and 1, another between 4 and 5, and two of them at 2. We have “touchdown” end behavior. So we have the following graph:
\[ f(x) = x^5 - 7x^4 + 5x^3 + x^2 - 6x \]

1. Fundamental Theorem: 5, 3, or 1 real zero.

2. Descartes' Rule of signs:

\[ f(x) = x^5 - 7x^4 + 5x^3 + x^2 - 6x \]

3 or 1 positive real zeros

\[ f(-x) = -x^5 - 7x^4 - 5x^3 + x^2 + 6x \]

1 negative real zero

Again, we have a case where there’s Descartes’ Rule says there should be 4 or 2 or 0 intercepts, but the Fundamental Theorem says there needs to be 5, 3 or 1. Again, the explanation is that \( f(0) = 0 \). Zero is neither positive nor negative, so Descartes' rule of Signs misses it completely.

3. I’d like to look for the negative real zero first, since there’s only one of them. So, I’ll set up my synthetic division, using positive 1 and \( f(-x) \).

\[
\begin{array}{cccccc}
1 & -1 & -7 & -5 & 1 & 6 & 0 \\
 & 1 & -8 & -13 & -12 & -6 \\
1 & -8 & -13 & -12 & -6 & -6 \\
\end{array}
\]

No sign changes means that we passed our zero. So we have found our single negative zero.

4. Positive zeroes:

\[
\begin{array}{cccccc}
1 & 1 & -7 & 5 & 1 & -6 & 0 \\
 & 1 & -6 & -1 & 0 & -6 \\
1 & -6 & -1 & 0 & -6 & -6 \\
\end{array}
\]

I see only one remaining sign change. So we’re only looking for one zero.
EASTER EGG:

If you look at the preceding synthetic division, you might notice something. You will always start with 1, then you will continue to get all negative numbers in your answer as long as you test any point smaller than 7 (because -7 is the second coefficient). As soon as you try 7:

\[
\begin{array}{c|cccc}
7 & 1 & -7 & 5 & 1 & -6 & 0 \\
 & 7 & 0 & 35 & 252 & + \\
\hline
1 & 0 & 5 & 36 & 246 & + \\
\end{array}
\]

So if \(x \geq 7\), you have no sign changes. If \(x = 6\), you'll get a negative second term. So I know my only remaining zero is between 6 and 7 without having to check 2,3,4,5, or 6.

End behavior: ballerina

Zeroes: between -1 and 0
0
between 6 and 7

Graph: